CS395T: Continuous Algorithms
Homework V

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Due date: April 11, 2024, start of class (3:30 PM).

Please list all collaborators on the first page of your solutions. Unless we have discussed and I have specified otherwise, homework is not accepted if it is not turned in by hand at the start of class, or turned in electronically on Canvas by then. Send me an email to discuss any exceptions.

1 Problem 1

Let $\pi(x) = \exp(-V(x))$ be a density on $\mathbb{R}^d$, i.e. with $\int \pi(x)\mathrm{d}x = \int \exp(-V(x))\mathrm{d}x = 1$, and suppose $V : \mathbb{R}^d \to \mathbb{R}$ is L-smooth and twice-differentiable. Prove that

$$\mathbb{E}_{x\sim\pi}[\|\nabla V(x)\|_2^2] \le Ld.$$

2 Problem 2

Let $\pi = \mathcal{N}(\mu, \mathbf{A})$, $\pi' = \mathcal{N}(\nu, \mathbf{B})$ be multivariate normal densities on $\mathbb{R}^d$, with respective means and full-rank covariance matrices $(\mu, \mathbf{A}) \in \mathbb{R}^d \times \mathbb{S}^{d\times d}_{\succ \mathbf{0}}$ and $(\nu, \mathbf{B}) \in \mathbb{R}^d \times \mathbb{S}^{d\times d}_{\succ \mathbf{0}}$.

(i) Brenier's theorem (see Proposition 4, Part XIII) states there is a unique optimal transport map $\nabla\varphi$ sending $x \sim \pi$ to $y = \nabla\varphi(x) \sim \pi'$, for convex $\varphi : \mathbb{R}^d \to \mathbb{R}$, such that $(x, \nabla\varphi(x))$ for $x \sim \pi$ is the coupling realizing $W_2^2(\pi, \pi')$. What is φ in the above setting?[1]

(ii) Compute $W_2^2(\pi, \pi')$ in the above setting.

3 Problem 3

The *underdamped Langevin dynamics* (ULD) is the drift-diffusion process on $\{(x_t, v_t)\}_{t\ge 0} \subset \mathbb{R}^d \times \mathbb{R}^d$ satisfying the following stochastic differential equation, for a parameter $\gamma > 0$:

$$\mathrm{d}\begin{pmatrix} x_t \\ v_t \end{pmatrix} = \begin{pmatrix} v_t \\ -v_t - \gamma\nabla V(x_t) \end{pmatrix}\mathrm{d}t + \begin{pmatrix} \mathbf{0}_d & \mathbf{0}_d \\ \mathbf{0}_d & \sqrt{2\gamma}\mathbf{I}_d \end{pmatrix}\mathrm{d}B_t,$$

where $\{B_t\}_{t\ge 0}$ is Brownian motion in $\mathbb{R}^{2d}$, and $V : \mathbb{R}^d \to \mathbb{R}$ satisfies $\int \exp(-V(x))\mathrm{d}x < \infty$.

(i) Let $\mathcal{L}$ be the generator of the ULD. Give a formula for $\mathcal{L}f$, for a smooth function $f : \mathbb{R}^{2d} \to \mathbb{R}$.

(ii) Let $\mathcal{L}^*$ be the adjoint of $\mathcal{L}$. Give a formula for $\mathcal{L}^*\pi$, for a probability density $\pi \in \mathcal{P}(\mathbb{R}^{2d})$.

(iii) Verify that

$$\pi^\star(x, v) \propto \exp\left(-V(x) - \frac{1}{2\gamma}\|v\|_2^2\right)$$

is a stationary distribution for the ULD.

[1] The identity $\mathbf{A}^{-\frac{1}{2}}(\mathbf{A}^{\frac{1}{2}}\mathbf{B}\mathbf{A}^{\frac{1}{2}})^{\frac{1}{2}}\mathbf{A}^{-\frac{1}{2}} = \mathbf{A}^{-1}(\mathbf{A}\mathbf{B})^{\frac{1}{2}}$ may be helpful.

4 Problem 4

(i) Let $\pi \in \mathcal{P}(\mathbb{R}^d)$, and let $f : \mathbb{R}^d \to \mathbb{R}$ satisfy $\int f(x)^2 \pi(x) \mathrm{d}x < \infty$. Prove that

$$\mathrm{Var}_\pi [f] = \inf_{m \in \mathbb{R}} \mathbb{E}_\pi \left[(f(x) - m)^2 \right].$$

(ii) Let $\mu, \pi \in \mathcal{P}(\mathbb{R}^d)$ have $c \leq \frac{\mu(x)}{\pi(x)} \leq C$ for all $x \in \mathbb{R}^d$, and let π satisfy a Poincaré inequality with constant C_{PI}. Prove that μ satisfies a Poincaré inequality with constant $\frac{C}{c} \cdot C_{\mathrm{PI}}$.

5 Problem 5

Let $\pi_0 \in \mathcal{P}(\mathbb{R}^d)$ be logconcave, and define a family of densities $\{\pi_t\}_{t \geq 0} \subset \mathcal{P}(\mathbb{R}^d)$ following the SDE

$$\mathrm{d}\pi_t(x) = \pi_t(x)(x - \mu_t)^\top \mathrm{d}B_t, \tag{1}$$

where $\{B_t\}_{t \geq 0}$ is Brownian motion in $\mathbb{R}^d$ adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$, and

$$\mu_t := \mathbb{E}_{x \sim \pi_t}[x] = \int x \pi_t(x) \mathrm{d}x, \text{ for all } t \geq 0.$$

Prove that π_t is t-strongly logconcave for all $t \geq 0$, regardless of the realization of $\mathcal{F}_t$.[2] You may use without proof that the family of $\{\pi_t\}_{t \geq 0}$ defined by (1) remain probability densities.

[2]It may be helpful to first compute $\mathrm{d} \log \pi_t(x)$. This problem defines the *stochastic localization* process, and shows that every measure can be decomposed as a mixture over t-strongly logconcave measures, for any $t \geq 0$.