Kevin's Favorite Inequality

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I was inspired by [Spi18] to describe my favorite "simple" inequality.

Fact 1. *Let* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$*, and let* $C > 0$*. Then,*

$$\|\mathbf{u}+\mathbf{v}\|_2^2 \le (1+C)\|\mathbf{u}\|_2^2 + \left(1+\frac{1}{C}\right)\|\mathbf{v}\|_2^2.$$

Proof. This follows by expanding both sides out, cancelling $\|\mathbf{u}\|_2^2 + \|\mathbf{v}\|_2^2$, and observing

$$2\langle \mathbf{u}, \mathbf{v}\rangle = 2\left\langle \sqrt{C}\mathbf{u}, \frac{1}{\sqrt{C}}\mathbf{v}\right\rangle \le C\|\mathbf{u}\|_2^2 + \frac{1}{C}\|\mathbf{v}\|_2^2.$$

□

When designing algorithms over $\mathbb{R}^d$, one of the most useful potential functions is a squared Euclidean norm. For example, it is differentiable and comparable to many functions via smoothness and strong convexity. One downside, though, is that it does not obey the triangle inequality. Fact 1 is an "approximate triangle inequality" for the squared Euclidean norm, and can often tightly characterize worst-case potential growths if $\|\mathbf{u}\|_2$, $\|\mathbf{v}\|_2$ are quite imbalanced.

References

[Spi18] Daniel A. Spielman. Dan's favorite inequality. https://www.cs.yale.edu/homes/spielman/561/lect03b-18.pdf, 2018. Accessed: 2025-04-10.