# On the Power Dominating Sets of Hypercubes

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## Abstract

*The performance of electrical networks is monitored by expensive Phasor Measurement Units (PMUs). It is economically beneficial to determine the optimal placement and the minimum number of PMUs required to effectively monitor an entire network. This problem has a graph theory model involving power dominating sets in a graph. A set $S$ of vertices in a graph is called a power dominating set if every vertex and every edge in the graph is "observed" by $S$ according to a set of observation rules. The power domination number of a graph is the minimum cardinality of a power dominating set of the graph. In this paper, the power domination number is determined for hypercubes $Q_n$ with $n = 2^k$, where $k$ is any positive integer.*

**Keywords:** Dominating set, power dominating set, zero forcing set, hypercube

## 1. Introduction

The performance of electrical networks is monitored by expensive Phasor Measurement Units (PMUs). It is economically beneficial to determine the optimal placement and the minimum number of PMUs required to effectively monitor an entire network. The network monitoring problem, as introduced in [3], asks for as few PMUs as possible to be put in the network system. In 2002, Haynes et al [8] formulated this network monitoring problem as a variation of the domination problem in graph theory.

We will follow the notation from [11]. All graphs $G = (V, E)$ considered are finite and simple. For a vertex $u$, the *open neighborhood* of $u$, denoted by $N(u)$, contains all neighbors of $u$ in $G$. The *closed neighborhood* of $u$, denoted by $N[u]$, contains all neighbors of $u$ as well as $u$ itself; that is, $N[u] = N(u) \cup \{u\}$. A set $S \subset V$ is a *dominating set* in $G$ if every vertex outside of $S$ has at least one neighbor in $S$. The *domination number* of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. The power domination problem is a variation of the domination problem. A set $S$ of vertices in $G$ is called a *power dominating set* if every vertex and every edge in $G$ is "observed" after repeatedly applying the following Observation Rules [8]:

0. Any vertex of $S$ is observed, and any edge incident to at least one vertex in $S$ is observed.
1. Any vertex that is incident to an observed edge is observed.
2. Any edge joining two observed vertices is observed.
3. If a vertex is incident to a total of $k > 1$ edges and if $k - 1$ of these edges are observed, then all $k$ of these edges are observed.

In other words, all observed vertices can be obtained from $S$ as follows. First all vertices in the closed neighborhood of $S$ are observed. Then repeatedly apply Observation Rule 3 to observe more and more new vertices until no new vertices in $G$ are observed. If the final set of observed vertices is $V(G)$, then $S$ is a power dominating set. The power domination number of $G$, denoted by $\gamma_p(G)$, is the minimum cardinality of a power dominating set of $G$.

It is known that the power domination number is NP-complete even when restricted to bipartite graphs and chordal graphs [8], to planar graphs and circle graphs [6], and to split graphs [6, 10]. On the other hand, the power domination problem has efficient linear time algorithms for trees [8], for graphs with bounded treewidth [9], for interval graphs if the interval ordering of the graph is provided [10], and for block graphs [12]. The power domination number of block graphs has been further studied in [2]. Upper bounds on the power domination number for a connected graph with at least three vertices and a connected claw-free cubic graph are presented in [13]. Closed formulas for the power domination number of grid graphs are obtained in [5], and formulas for

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978-0-7695-4477-9/11 $26.00 © 2011 IEEE
DOI 10.1109/CSE.2011.211

direct product and strong product of path graphs are obtained in [4]. In this paper, we study the power domination problem for hypercube $Q_n$ and prove the following theorem.

**THEOREM 1.**

$$\frac{2^{n-1}}{n} \leq \gamma_p(Q_n) \leq 2^{n-\lfloor \log_2 n \rfloor -1}.$$

*In particular, if $n = 2^k$ for some positive integer $k$, then*

$$\gamma_p(Q_n) = 2^{n-k-1}.$$

In Section 2, we study the domination problem for hypercubes. In Section 3, we find a connection between a power dominating set and a zero forcing set in a graph. Theorem 1 is proved in Section 4. Finally a conjecture on the power domination number of hypercubes is presented.

## 2. Domination of Hypercubes

The vertex set of the $n$-dimensional hypercube $Q_n$ is $V(Q_n) = \{(a_1, \ldots, a_n) : a_i = 0 \text{ or } 1\}$, and two vertices $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ in $Q_n$ are adjacent iff $\sum_i |a_i - b_i| = 1$. We may also view each vertex of $Q_n$ as an $n$-dimensional vector over $\mathbb{F}_2$. Thus throughout this section all matrix addition and multiplication are performed over $\mathbb{F}_2$, that is, $1 + 1 = 0$. We use rank $A$ to denote the rank of a matrix $A$ over $\mathbb{F}_2$, and use dim $S$ to denote the dimension of a vector space $S$ over $\mathbb{F}_2$.

**Example 1.** The matrix equation

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has the solution set $\{(0,0,0)^T, (1,1,1)^T\}$ which corresponds to a two-vertex dominating set for $Q_3$.

**Example 2.** The matrix equation

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has the solution set

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

which corresponds to a dominating set of $16$ $(= 2^4)$ vertices for $Q_7$.

**LEMMA 1.** *If $n = 2^k - 1$ for some positive integer $k$, then $\gamma(Q_n) = 2^n/(n+1) = 2^{n-k}$.*

The lemma was mentioned without proof in [7, Page 279]. We provide the following proof to make this paper self-contained.

*Proof.* Recall that the number of non-empty subsets of $\{1, \ldots, k\}$ is exactly $n$ $(= 2^k - 1)$. So, as suggested by Examples 1 and 2, we may define $A$ to be the $k$ by $n$ matrix such that the $n$ columns of $A$ are the indicator vectors of all $n$ non-empty subsets of $\{1, \ldots, k\}$. We consider the matrix equation

$$A\mathbf{x} = \mathbf{0},$$

where $\mathbf{x} \in \mathbb{F}_2^n$ and $\mathbf{0}$ is the all 0's vector in $\mathbb{F}_2^k$. (See Examples 1 and 2 for the cases $n = 3$ and 7, respectively.) Let $S$ be the solution set of the above matrix equation. Then

$$\dim S = n - \text{rank } A = n - k,$$

and thus $|S| = 2^{n-k}$. We want to show that $S$ is a minimum dominating set for $Q_n$.

Claim 1: Any non-zero vector $\mathbf{x}$ in $S$ has at least three non-zero entries.

Proof of Claim 1. Since $A$ has no zero column, no $\mathbf{x}$ in $S$ has exactly one non-zero entry. Since all columns of $A$ are distinct, no $\mathbf{x}$ in $S$ has exactly two non-zero entries.

Claim 2. Any two vectors in $S$ has Hamming distance at least three.

Proof of Claim 2. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be any two distinct vectors in $S$. Then

$$A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{0},$$

and thus $(\mathbf{x}_1 - \mathbf{x}_2)$ is a non-zero vector in $S$. By Claim 1, $(\mathbf{x}_1 - \mathbf{x}_2)$ has at least three non-zero entries; that is, $\mathbf{x}_1$ and $\mathbf{x}_2$ have Hamming distance at least three.

Claim 3. $S$ is a dominating set of $Q_n$.

Proof of Claim 3. By Claim 2, $N[\mathbf{x}_1] \cap N[\mathbf{x}_1] = \emptyset$ for any distinct $\mathbf{x}_1$ and $\mathbf{x}_2$ in $S$. Thus

$$\begin{aligned} |N[S]| := |\cup_{\mathbf{x} \in S} N(\mathbf{x})| &= \textstyle\sum_{\mathbf{x} \in S} |N(\mathbf{x})| = (1 + d(\mathbf{x}))|S| \\ &= (n+1)2^{n-k} = 2^n = |V(Q_n)|, \end{aligned}$$

which implies that $S$ is a dominating set of $Q_n$.

Claim 4. $S$ is a minimum dominating set of $Q_n$.

Proof of Claim 4. Since each vertex in $Q_n$ can dominate $(n+1)$ vertices, any dominating set of $Q_n$ contains at leave $2^n/(n+1)$ $(= 2^{n-k})$ vertices. Since $|S| = 2^{n-k}$, Claim 4 follows from Claim 3. Therefore

$$\gamma(Q_n) = |S| = 2^{n-k}.$$

□

## 3. Zero Forcing Set

In this section, we derive a connection between a power dominating set and a zero forcing set in a graph. To define a zero forcing set in $G$, one needs to start with the following Color-Change Rule [1].

**Color-Change Rule.** Let each vertex of $G$ be colored either white or black. If a vertex $u$ is a black vertex of $G$ and exactly one neighbor $v$ of $u$ is white, then change the color of $v$ to black.

A vertex set $Z \subset V$ of $G$ is called a *zero forcing set* if, initially all vertices in $Z$ are colored black and the remaining vertices in $V \setminus Z$ are colored white, then after repeatedly applying the above Color-Change Rule the derived coloring of $G$ is all black. The zero forcing number, denoted by $Z(G)$, is the minimum cardinality of a zero forcing set $Z$ in $G$. The following lemma reveals a connection between a power dominating set and the zero forcing number in a graph.

**LEMMA 2.** *Let $\{u_1, \ldots, u_t\}$ be a power dominating set for a graph $G$ with no isolated vertices. Then*

$$Z(G) \leq \sum_{i=1}^{t} d(u_i).$$

*Proof.* Since $\{u_1, \ldots, u_t\}$ is a power dominating set for $G$, by the definition of power domination, $\cup_{i=1}^{t} N[u_i]$ is a zero forcing set for $G$.

We may assume without loss of generality that $N(u_i) - \{u_1, \ldots, u_t\} \neq \emptyset$ for each $i$ with $1 \leq i \leq t$. (Otherwise, if $N(u_i) \subseteq \{u_1, \ldots, u_t\}$ for some $i$, then $(\{u_1, \ldots, u_t\} - u_i)$ would be a smaller power dominating set for $G$ and thus one can prove the lemma on this smaller set instead.) For each $u_i$, choose $v_i \in (N(u_i) - \{u_1, \ldots, u_t\})$. Now we want to prove that the following set

$$Z := \cup_{i=1}^{t} \left(N[u_i] - v_i\right)$$

is a zero forcing set for $G$.

Initially all vertices in $Z$ are colored black and the remaining vertices in $V \setminus Z$ are colored white. Note that

$$(N[u_i] - v_i) \subseteq N[u_i] \cap Z \subseteq N[u_i].$$

So either $N[u_i] \cap Z = N[u_i] - v_i$ or $N[u_i] \cap Z = N[u_i]$. If $N[u_i] \cap Z = N[u_i] - v_i$, then $u_i$ was colored black and exactly one neighbor, $v_i$, of $u_i$ was colored white. By applying the Color-Change Rule to $u_i$, the vertex $v_i$ will changes its color from white to black. So by applying the Color-Change Rule to each $u_i$ if necessary, the set of black vertices can be extended from $Z$ to $Z \cup \{v_1, \ldots, v_t\} = \cup_{i=1}^{t} N[u_i]$. Since $\cup_{i=1}^{t} N[u_i]$ is a zero forcing set, the set $Z$ is a zero forcing set as well. Thus

$$\begin{aligned} Z(G) &\leq |\cup_{i=1}^{t} \left(N[u_i] - v_i\right)| \\ &\leq \textstyle\sum_{i=1}^{t} |N[u_i] - v_i| = \sum_{i=1}^{t} d(u_i). \end{aligned}$$

□

**THEOREM 2.**

$$\gamma_p(Q_n) \geq \frac{2^{n-1}}{n}.$$

*Proof.* Let $t := \gamma_p(Q_n)$ and $\{u_1, \ldots, u_t\}$ be a power dominating set for $Q_n$. By Lemma 2,

$$Z(Q_n) \leq \sum_{i=1}^{t} d(u_i) = n \cdot \gamma_p(Q_n).$$

By [1, Theorem 3.1], we have $Z(Q_n) = 2^{n-1}$. Thus

$$\gamma_p(Q_n) \geq \frac{Z(Q_n)}{n} = \frac{2^{n-1}}{n}.$$

□

## 4. Power Domination of Hypercubes

In this section we will prove Theorem 1. The proof follows from Theorem 2 and the next theorem.

**THEOREM 3.** *Let $Q_n$ be the $n$-dimensional hypercube. Then*

$$\gamma_p(Q_n) \leq 2^{n - \lfloor \log_2 n \rfloor - 1}.$$

*Proof.* Let $k := \lfloor \log_2 n \rfloor$ and $m := 2^k - 1$. By Lemma 1,

$$\gamma(Q_m) = 2^{m-k}.$$

Recall that $n - 1 \geq 2^k - 1 = m$. By an easy induction argument, one can prove that

$$\gamma\left(Q_{n-1}\right) \leq 2^{n-m-1}\gamma(Q_m) = 2^{n-k-1}.$$

By Observation Rule 3, any dominating set for a copy of $Q_n$ is always a power dominating set for $Q_{n+1}$. Thus one has $\gamma_p(Q_{n+1}) \leq \gamma(Q_n)$ and thus

$$\gamma_p(Q_n) \leq \gamma\left(Q_{n-1}\right) \leq 2^{n-k-1}.$$

□

In the proof of the above theorem, we use the observation $\gamma_p(Q_{n+1}) \leq \gamma(Q_n)$. We conjecture that any minimum dominating set for $Q_n$ always corresponds to a minimum power dominating set for $Q_{n+1}$.

**CONJECTURE 1.**

$$\gamma_p(Q_{n+1}) = \gamma(Q_n).$$

**Acknowledgement.** Some initial work of this paper was done in a research project by three high school students (Ilic, Ramirez, and Tian) in the summer of 2010 under the supervision of Dr. Nathaniel Dean at Texas State University. Ilic, Ramirez, and Tian's research project won them spots as regional finalists in the 2010 Siemens competition in mathematics, science, and technology. We thank Texas State Honors Math Camp for providing this faculty-and-student mentoring/research opportunity.

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